Linear Algebra I

30/01/2014, Thursday, 9:00-12:00

You are **NOT** allowed to use any type of calculators.

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1 (7+8=15 pts) **Linear equations**

Three people play a game in which there are always two winners and one looser. They have the understanding that the loser gives each winner an amount equal to what the winner already has. After three games, each has lost just once and each has €24.

(a) Let x_k be the amount of money Player k begins with. Assume that Player 1 has lost the first game, Player 2 the second game, and Player 3 the third game. Write down three equations in terms of x_1, x_2, and x_3 for the amount of money each has after three games.

(b) Solve the linear equations obtained above to determine x_1, x_2, and x_3.

REQUIRED KNOWLEDGE: **linear equations, Gauss-elimination, row reduced echelon form.**

SOLUTION:

1a: After the first round of game, the players will have

$$\begin{aligned}
&\text{Player 1:} && x_1 - (x_2 + x_3) = x_1 - x_2 - x_3 \\
&\text{Player 2:} && x_2 + x_2 = 2x_2 \\
&\text{Player 3:} && x_3 + x_3 = 2x_3
\end{aligned}$$

as Player 1 has lost the round. After the second round, they will have

$$\begin{aligned}
&\text{Player 1:} && (x_1 - x_2 - x_3) + (x_1 - x_2 - x_3) = 2x_1 - 2x_2 - 2x_3 \\
&\text{Player 2:} && 2x_2 - (x_1 - x_2 - x_3) - 2x_3 = -x_1 + 3x_2 - x_3 \\
&\text{Player 3:} && 2x_3 + 2x_3 = 4x_3
\end{aligned}$$

and after the third

$$\begin{aligned}
&\text{Player 1:} && (2x_1 - 2x_2 - 2x_3) + (2x_1 - 2x_2 - 2x_3) = 4x_1 - 4x_2 - 4x_3 \\
&\text{Player 2:} && (-x_1 + 3x_2 - x_3) + (-x_1 + 3x_2 - x_3) = -2x_1 + 6x_2 - 2x_3 \\
&\text{Player 3:} && 4x_3 - (2x_1 - 2x_2 - 2x_3) - (-x_1 + 3x_2 - x_3) = -x_1 - x_2 + 7x_3
\end{aligned}$$

as the losers are respectively, Player 2 and 3. As they all have €24 after three rounds, we arrive at the linear equation:

$$\begin{aligned}
4x_1 - 4x_2 - 4x_3 &= 24 \\
-2x_1 + 6x_2 - 2x_3 &= 24 \\
-x_1 - x_2 + 7x_3 &= 24.
\end{aligned}$$

1b: We first form the augmented matrix:

$$\begin{pmatrix} -1 & -1 & 7 & \vdots & 24 \\ -2 & 6 & -2 & \vdots & 24 \\ 4 & -4 & -4 & \vdots & 24 \end{pmatrix}$$

Then, we perform row operations to put the augmented matrix into the *reduced row echelon form*:

$$\left(\begin{array}{ccc|c} -1 & -1 & 7 & 24 \\ -2 & 6 & -2 & 24 \\ 4 & -4 & -4 & 24 \end{array}\right) \xrightarrow{\textbf{1st}=-1\times\textbf{1st}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ -2 & 6 & -2 & 24 \\ 4 & -4 & -4 & 24 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ -2 & 6 & -2 & 24 \\ 4 & -4 & -4 & 24 \end{array}\right) \xrightarrow{\textbf{2nd}=2\times\textbf{1st}+\textbf{2nd}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 4 & -4 & -4 & 24 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 4 & -4 & -4 & 24 \end{array}\right) \xrightarrow{\textbf{3rd}=-4\times\textbf{1st}+\textbf{3rd}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right) \xrightarrow{\textbf{2nd}=\frac{1}{8}\times\textbf{2nd}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & -8 & 24 & 120 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & -8 & 24 & 120 \end{array}\right) \xrightarrow{\textbf{3rd}=\frac{1}{8}\times\textbf{3rd}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & 3 & 15 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & 3 & 15 \end{array}\right) \xrightarrow{\textbf{3rd}=\textbf{2nd}+\textbf{3rd}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{\textbf{2nd}=2\times\textbf{3rd}+\textbf{2nd}} \left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{\textbf{1st}=7\times\textbf{3rd}+\textbf{1st}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 60 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 60 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{\textbf{1st}=-1\times\textbf{2nd}+\textbf{1st}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right).$$

There are no free variables and hence the solution is unique: $x_1 = 39$, $x_2 = 21$, and $x_3 = 12$.

Let

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

where all four blocks are $n \times n$ matrices.

(a) If A and B are nonsingular matrices, show that M must also be nonsingular and that M^{-1} must be of the form

$$M^{-1} = \begin{bmatrix} A^{-1} & D \\ 0 & B^{-1} \end{bmatrix}$$

for some $n \times n$ matrix D.

(b) Determine D.

REQUIRED KNOWLEDGE: **nonsingular matrices, inverse of a matrix, partitioned matrices.**

SOLUTION:

2a: Let $z \in \mathbb{R}^{2n}$ be a vector such that $Mz = 0$. Partition z as follows:

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

where $x, y \in \mathbb{R}^n$. Then, we have

$$0 = Mz = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + Cy \\ By \end{bmatrix}.$$

Hence, we get $Ax + Cy = By = 0$. It follows from nonsingularity of B that $y = 0$. Then, we get $Ax = 0$. Since A is nonsingular, this can happen only if $x = 0$. Therefore, we have $z = 0$. Consequently, M is nonsingular.

Let the inverse of M be given by

$$M^{-1} = \begin{bmatrix} U & W \\ X & Y \end{bmatrix}$$

where all four blocks are $n \times n$ matrices. Note that

$$I = MM^{-1} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} U & W \\ X & Y \end{bmatrix} = \begin{bmatrix} AU + CX & AW + CY \\ BX & BY \end{bmatrix}.$$

This yields the following equations:

$$\begin{aligned} AU + CX &= I \\ BX &= 0 \\ BY &= I. \end{aligned}$$

Since B is nonsingular, we get $X = 0$, $Y = B^{-1}$, and $AU = I$. Since A is nonsingular, the latter results in $U = A^{-1}$. Therefore, M^{-1} is of the form

$$M^{-1} = \begin{bmatrix} A^{-1} & D \\ 0 & B^{-1} \end{bmatrix}$$

for some matrix D.

2b: Note that

$$I = MM^{-1} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} A^{-1} & D \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} I & AD + CB^{-1} \\ 0 & I \end{bmatrix}.$$

This yields $AD + CB^{-1} = 0$. By multiplying from left by A^{-1}, we obtain $D = -A^{-1}CB^{-1}$.

Let M be an 4×4 matrix with the characteristic polynomial $p_M(\lambda) = \lambda(\lambda - 1)(\lambda^2 + 1)$.

(a) Is M nonsingular? Explain.

(b) Determine the eigenvalues of M.

(c) Determine the determinant of M.

(d) Determine the rank of M.

(e) Is M diagonalizable? Explain.

REQUIRED KNOWLEDGE: **nonsingularity, eigenvalues, determinant, rank, diagonalizability.**

SOLUTION:

3a: A square matrix is singular if and only if one of its eigenvalues is zero. Since M has a zero eigenvalue, it has to be singular.

3b: Eigenvalues of a square matrix are the roots of its characteristic polynomial. By solving $p_M(\lambda) = 0$, we get $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_{3,4} = \pm i$ as eigenvalues.

3c: The determinant is equal to the product of the eigenvalues. Then, we get $\det(M) = 0$.

3d: Every eigenvector corresponding a nonzero eigenvalue belong to the column space of the matrix. Since eigenvectors corresponding to distinct eigenvalues are linearly independent and M has three distinct nonzero eigenvalue, there must at least three linearly independent vectors in the column space of M. Then, we have

$$\operatorname{rank}(M) = \dim(\mathcal{R}(M)) \geqslant 3.$$

Since every eigenvctor corresponding to the zero eigenvalue must belong to the null space of M, we have

$$\operatorname{null}(M) = \dim(\mathcal{N}(M)) \geqslant 1.$$

From the rank-nullity theorem, we know that

$$\operatorname{rank}(M) + \operatorname{null}(M) = 4.$$

Together with the above inequalities, this implies that

$$\operatorname{rank}(M) = 3 \qquad \text{and} \qquad \operatorname{null}(M) = 1.$$

3e: Since M has distinct eigenvalues, it is diagonalizable.

Consider the vector space of $k \times k$ matrices, i.e. $\mathbb{R}^{k\times k}$. Let N_k denote the set of all nonsingular $k \times k$ matrices, S_k denote the set of all $k \times k$ symmetric matrices, and T_k denote the set of all $k \times k$ skew-symmetric matrices.

(a) Show that N_k is *not* a subspace of $\mathbb{R}^{k\times k}$. Show that both S_k and T_k are subspaces of $\mathbb{R}^{k\times k}$.

(b) Determine a basis for S_3 as well as its dimension. What would be the dimension of S_k?

(c) Determine a basis for T_3 as well as its dimension. What would be the dimension of T_k?

(d) Show that $\mathbb{R}^{k\times k}$ is the direct sum of S_k and T_k, that is $\mathbb{R}^{k\times k} = S_k \oplus T_k$.

(e) Let $L : \mathbb{R}^{k\times k} \to \mathbb{R}^{k\times k}$ be the transformation given by $L(A) = \dfrac{A + A^T}{2}$. Show that L is a linear transformation.

Hint: A square matrix M is called *skew-symmetric* if $M = -M^T$.

REQUIRED KNOWLEDGE: **vector spaces, subspaces, basis, dimension, direct sum, linear transformations.**

SOLUTION:

4a: Both I_k and $-I_k$ are nonsingular and hence in N_k, but $I_k + (-I_k) = 0$ is not nonsingular. This means that the set of nonsingular matrices N_k is *not* closed under matrix addition. As such, N_k is *not* a subspace.

Clearly, the zero matrix is symmetric and skew-symmetric at the same time. This means that both S_k and T_k are not empty sets. Since $(A+B)^T = A^T + B^T$ and $(\alpha A) = \alpha A^T$ for all matrices A, B and scalars α. As such, we can conclude that both S_k and T_k are closed under matrix addition and scalar multiplication. Consequently, they are both subspaces.

4b: A matrix M belongs to S_3 if and only if it is of the form

$$M = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

for some real numbers a, b, c, d, e, and f. Then, we have

$$M = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we can conclude that the six matrices above span S_3. Note that $M = 0$ if and only if $a = b = c = d = e = f = 0$. As such, these six matrices are linearly independent. Consequently, they form a basis for S_3 and $\dim(S_3) = 6$.

The argument above suggests that the dimension of the subspace S_k would be the number of diagonal elements plus half of the number of the off-diagonal elements, that is

$$\dim(S_k) = k + \frac{k^2 - k}{2} = \frac{k^2 + k}{2} = \frac{k(k+1)}{2}.$$

4c: A matrix M belongs to T_3 if and only if it is of the form

$$M = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

for some real numbers a, b, c, d, e, and f. Then, we have

$$M = a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

In a similar fashion to the problem 4b, we can conclude that these three matrices form a basis and hence $\dim(T_3) = 3$. In addition, the dimension of the subspace T_k would be half of the number of the off-diagonal elements, that is

$$\dim(T_k) = \frac{k^2 - k}{2} = \frac{k(k-1)}{2}.$$

4d: In order to prove that $\mathbb{R}^{k\times k} = S_k \oplus T_k$, one needs to show that every $k \times k$ matrix M can be written as a unique sum of two matrices $P \in S_k$ and $Q \in T_k$. For a given matrix M, let

$$P = \frac{M + M^T}{2} \quad \text{and} \quad Q = \frac{M - M^T}{2}.$$

Note that

$$P^T = (\frac{M + M^T}{2})^T = \frac{M^T + M}{2} = P \quad \text{and} \quad Q^T = (\frac{M - M^T}{2})^T = \frac{M^T - M}{2} = -Q.$$

In other words, $P \in S_k$ and $Q \in T_k$. Observe that

$$P + Q = \frac{M + M^T}{2} + \frac{M - M^T}{2} = M.$$

Therefore, we proved that every matrix can be written as a sum of a symmetric and a skew-symmetric matrix. To prove that this sum is unique. Let $P' \in S_k$ and $Q' \in T_k$ be such that

$$M = P' + Q'.$$

Then, we have $P + Q = P' + Q'$. Equivalently, $P - P' = Q' - Q$. Since both S_k and T_k are subspaces, we can conclude that $P - P' = Q' - Q \in S_k \cap T_k$. In other words, the matrix $P - P'$ is symmetric and skew-symmetric at the same time. This is possible only if $P = P'$ and hence $Q = Q'$.

4e: Note that

$$L(A + B) = \frac{(A + B) + (A + B)^T}{2} = \frac{A + A^T}{2} + \frac{B + B^T}{2} = L(A) + L(B)$$

for any matrices A, $B \in \mathbb{R}^{k\times k}$.

Note also that

$$L(\alpha A) = \frac{\alpha A + (\alpha A)^T}{2} = \frac{\alpha A + \alpha A^T}{2} = \alpha\frac{A + A^T}{2} = \alpha L(A)$$

for any matrix $A \in \mathbb{R}^{k\times k}$ and real number α.

Therefore, L is a linear transformation.

A *tridiagonal* matrix is a square matrix that has nonzero elements only on the main diagonal, the first diagonal below the main diagonal, and the first diagonal above the main diagonal. Consider the sequence of tridiagonal matrices $A_n \in \mathbb{R}^{n\times n}$ given by:

$$A_1 = 3,\ A_2 = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix},\ A_3 = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix},\ A_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix},\ A_5 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}, \ldots$$

Let $d_n := \det(A_n)$.

(a) Verify that $d_1 = 3$ and $d_2 = 7$.

(b) By using cofactor expansion, determine the numbers p and q such that $d_k = pd_{k-1} + qd_{k-2}$ for all $k \geqslant 3$.

(c) Note that $\begin{bmatrix} d_k \\ d_{k-1} \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{k-1} \\ d_{k-2} \end{bmatrix}$ for all $k \geqslant 3$ where p and q are as obtained above. Define $M = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$. Show that $d_k = \begin{bmatrix} 1 & 0 \end{bmatrix} M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$ for all $k \geqslant 3$.

(d) Determine a nonsingular matrix X and a diagonal matrix D such that $X^{-1}MX = D$. Determine M^k for all $k \geqslant 0$. Determine d_k for all $k \geqslant 1$.

Required Knowledge: **determinants, diagonalization.**

Solution:

5a: By definition, we have

$$d_1 = \det(A_1) = \det(3) = 3 \qquad \text{and} \qquad d_2 = \det(A_2) = \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\right) = 3\cdot 3 - 1\cdot 2 = 7.$$

5b: Note that

$$\begin{aligned} d_k &= \det(A_k) \\ &= 3\det(A_{k-1}) - \det\left(\left[\begin{array}{c|c} 2 & 1\ 0\ \cdots\ 0 \\ \hline 0_{k-2} & A_{k-2} \end{array}\right]\right) \qquad \text{[Here } 0_{k-2} \text{ denotes the zero vector of } \mathbb{R}^{k-2}\text{]} \\ &= 3d_{k-1} - 2\det(A_{k-2}) \\ &= 3d_{k-1} - 2d_{k-2}. \end{aligned}$$

for all $k \geqslant 3$. Therefore, $p = 3$ and $q = -2$.

5c: Note that

$$\begin{bmatrix} d_3 \\ d_2 \end{bmatrix} = M \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_4 \\ d_3 \end{bmatrix} = M \begin{bmatrix} d_3 \\ d_2 \end{bmatrix} = M^2 \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}.$$

By repeating the same argument above, we obtain

$$\begin{bmatrix} d_k \\ d_{k-1} \end{bmatrix} = M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$$

for all $k \geqslant 3$. By pre-multiplying by $\begin{bmatrix} 1 & 0 \end{bmatrix}$, we obtain

$$d_k = \begin{bmatrix} 1 & 0 \end{bmatrix} M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}.$$

5d: Note that

$$\det(M-\lambda I)=\det\left(\begin{bmatrix}3-\lambda & -2\\ 1 & -\lambda\end{bmatrix}\right)=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2).$$

Therefore, the eigenvalues of the matrix M are $\lambda_1=1$ and $\lambda_2=2$. To compute eigenvectors, we should solve

$$\begin{aligned}0&=(M-\lambda_1 I)x_1=\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix}x_1\\ 0&=(M-\lambda_2 I)x_2=\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix}x_2.\end{aligned}$$

These would yield, for instance,

$$x_1=\begin{bmatrix}1\\1\end{bmatrix}\qquad\text{and}\qquad x_2=\begin{bmatrix}2\\1\end{bmatrix}.$$

Let X be defined by

$$X:=\begin{bmatrix}x_1 & x_2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}.$$

Note that

$$X^{-1}=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$$

and also that

$$X^{-1}MX=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}=D.$$

Since $M=XDX^{-1}$, we know that $M^k=XD^kX^{-1}$. Hence, we get

$$M^k=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1^k & 0\\ 0 & 2^k\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}2^{k+1}-1 & 2-2^{k+1}\\ 2^k-1 & 2-2^k\end{bmatrix}$$

for all $k\geqslant 0$. Note that

$$\begin{aligned}d_k&=\begin{bmatrix}1 & 0\end{bmatrix}M^{k-2}\begin{bmatrix}d_2\\ d_1\end{bmatrix}\\ &=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}2^{k-1}-1 & 2-2^{k-1}\\ 2^{k-2}-1 & 2-2^{k-2}\end{bmatrix}\begin{bmatrix}7\\ 3\end{bmatrix}\\ &=\begin{bmatrix}2^{k-1}-1 & 2-2^{k-1}\end{bmatrix}\begin{bmatrix}7\\ 3\end{bmatrix}\\ &=7(2^{k-1}-1)+3(2-2^{k-1})\\ &=4\cdot 2^{k-1}-1\\ &=2^{k+1}-1.\end{aligned}$$

for all $k\geqslant 3$. Since $d_1=3$ and $d_2=7$, we have $d_k=2^{k+1}-1$ for all $k\geqslant 1$.

Given the data

x	-1	0	1	2
y	1.5	4.5	6.5	7.5

determine a line $y = ax + b$ that best fits the data in the least squares sense.

REQUIRED KNOWLEDGE: **least-squares problem.**

SOLUTION:

From the table, we obtain the linear equations

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1.5 \\ 4.5 \\ 6.5 \\ 7.5 \end{bmatrix}.$$

The solution of the least-squares problem can be found by solving the following equation:

$$\begin{aligned}
\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 4.5 \\ 6.5 \\ 7.5 \end{bmatrix} \\
\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -1.5 + 6.5 + 15 \\ 1.5 + 4.5 + 6.5 + 7.5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \\
\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 10 \\ 10 \end{bmatrix} \\
\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 10 \end{bmatrix} \\
\begin{bmatrix} a \\ b \end{bmatrix} &= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} \\
\begin{bmatrix} a \\ b \end{bmatrix} &= \frac{1}{5} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.
\end{aligned}$$